On Two Ways of Saying "No"

Classical and Intuitionist Negation in Mathematics

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Consider the simplest possible judgment, the atomic "S is p" in which a property p is attributed to an object S ("judgment" here refers to the content of an assertion, that which the assertion asserts, but also to the assertion itself, the utterance in which the judgment as content is asserted, or still the linguistic expression that carries the judgment; these distinctions are important, but I'll not make too much of them here). There are two ways in which this judgment can be negated, two different negations of "S is p", that is, the *internal* "S is not-p", where a negative property occurs and the *external* "not-(S is p)", where negation acts as a propositional connective. This distinction bears on questions of meaningfulness and meaninglessness. Consider, for example, the judgment "virtue is triangular" (Haack 1978: 35), a meaningless assertion due to the material incongruity of its terms; virtue and triangularity are *in principle* not fit for one another (I'll come back to this issue later). For the same reason the internal negation "virtue is not triangular", i.e. "virtue is non-triangular", is meaningless too. But, some claim, the external negation "it's not the case that virtue is triangular", i.e. "not-(virtue is triangular)" is not only meaningful but actually true. I'll pass over this problem in silence, however, since I'll be concerned here only with the external, propositional negation.

The meaning of negation is naturally tied up with the meaning of negative judgments. Under one interpretation, which I'll call the classical, the meaning of a judgment is given by the conditions under which it is true[1]. To know the meaning of $\neg$A (i.e. not-A) is, then, to know what must be the case for it to be true[2]:

(N) $\neg$A is true if, and only if, A is false. (Here, of course, we are supposing there are only two truth values, the true and the false. In case there are more (many-valued logics) there are many different non-equivalent notions of negation.)

So, for $\neg$A to be true it must be the case that what must be the case for A to be true is not the case. The alternative advanced by the logical and mathematical revisionists we call intuitionists is that meaning is not determined by truth conditions but by assertability conditions: to know the meaning of $\neg$A is to know on which conditions it can be asserted (as you can see there is a change here from judgment as content to judgment as assertion). One way of stating assertability conditions is by establishing *rules* by which judgments can be asserted provided other judgments have already been asserted. Rules that have a purely syntactic character, depending only on the logical form of the judgments in question and so, according to some, more appropriate as means of characterizing the meaning of logical constants (see Martin-Löf 96). The meaning of negation in particular is, in this case, given by rules determining which assertions follow from negated assertions (rules of elimination) and which

[1] Is meaning determined by the conditions under which the judgment *is* true or those under which it can be *asserted*? Of course, this has to do with whether we take judgments as propositions (units of meaning) or assertions involving a personal commitment on the part of who makes the assertion to that which is asserted (see Martin-Löf 1996).

[2] Here, we face the temptation of thinking that to know or have a grasp on what must be the case for A to be true is to have a "mental picture" of the situation "depicted" by A. This is an error, for if the meaning of A was a mental picture, considering the notion of mental picture unproblematic, which it isn't, meaning would be a private possession, and there would be no *objective* criterion for telling one who *really* grasps a meaning from one who only *believes* to be grasping a meaning.

assertions negated assertions follow from (rules of introduction). Of course, these rules are teleologically oriented at the preservation of truth (for we want to assert everything that is true but only what is true); so, they must be justified from the point of view of an intended semantics and a subjacent notion of truth.

But why should one prefer the intuitionist not the classical conception of meaning? Because, some argue (among them Michael Dummett, a modern intuitionist) mathematics, at least, requires it. The theory that to grasp the meaning of a judgment is to grasp its truth conditions, Dummett says, goes against standard mathematical practices. There are after all judgments mathematicians are willing to accept as true, for example "$\sqrt{2}^{\sqrt{2}}$ is rational *or* $\sqrt{2}^{\sqrt{2}}$ is irrational", without having any grasp on what makes them true, that is, in our example, which leg of the disjunction is true and why (we know today that $\sqrt{2}^{\sqrt{2}}$ is actually irrational, but the classical truth of the disjunction does not depend upon us knowing that). So, Dummett argues, if to grasp the meaning of a judgment is to grasp what must be the case for it to be true, how can one grasp the meaning of a judgment and not be able to recognize that that which makes it true is obtained, when it is obtained? So, if the meaning of judgments and, by extension, logical connectives, is to be given in terms of truth conditions we must, intuitionists claim, to be able to recognize that a truth condition is obtained when it is obtained.

Dummett believed, with Wittgenstein, that meaning is determined by use, which must be publically displayable (meaning cannot be a private possession). We must, he says, be able to somehow *exhibit* our grasp of the meaning of judgments. One way of so doing is by abiding to rules of assertability (to follow a rule, as Wittgenstein has argued, cannot ever be a private business) but certainly *not*, he claims, by stating truth-conditions we cannot always recognize as obtaining (see Dummett 1980: 215-47)[3]. Any justification for adopting intuitionist logic as the correct logic for mathematics must, according to Dummett, start with the thesis that meaning is exhaustively determined by use. In addition, Dummett claims, it must subscribe to a non-holistic conception of judging, one that accords to each judgment its own individual content. For, in this case, each judgment has its own individual meaning which, Dummett reasons, cannot be publically displayed if meaning is given by truth conditions we cannot always recognize as obtaining. If we insist that meaning is determined by truth conditions, we must be able to *display* our ability to recognize that truth conditions obtain when they obtain. This can only be done, or so the argument goes, if we can provide effective means of *verification* one can set in action if and when one so cares. But, as is clear, procedures of verification are as much conditions of assertability as conditions of truth: one can assert A provided one knows how to verify it. In the end, meaning, as intuitionists want it, is given by conditions of assertability.

So, everything boils down to this: for intuitionists, the meaning of a judgment is in the effective procedure for verifying it. If no such method exists, the judgment is devoid of meaning. Adopting such a conception has dramatic consequences for mathematics: a disjunction A or B, for instance, is meaningful only if we have the means for actually verifying either A or B. Hence, if neither A nor not-A can be verified the disjunction A or not-A ($A \vee \neg A$) is meaningless, we cannot assert it, it is neither true nor false, and there goes the principle of *tertium non datur*, a fact that has obvious consequences for the notion of negation and, consequently, for the whole of

[3] Of course, to state something is also a public display, but how can the grasp of the meaning of statements expressing our grasp of meanings be displayed? If by another statement we risk falling into an infinite regress or vicious circle. How, the intuitionist wonders, can one display one's grasp that a truth condition has obtained that is *not* by *stating* this fact? Only, he thinks, by providing publically accessible means for the *verification* that it has indeed obtained: instead of *saying* one *shows*.

mathematics (a well-known phenomenon associated with the intuitionist weakening of negation is the splitting of mathematical notions, see for instance Brouwer 1923, 1925, 1927).

Since intuitionists and classicists can agree, at least on a superficial level, that the meaning of negation is given by N both sides apparently give negation the same meaning, against Quine's claim that negation does not mean the same thing for the intuitionist as for the classicist. However, since they disagree on which the conditions for attributing truth and falsity are we can equally say they disagree on the matter. Classicists as well as intuitionists, however, agree on the following:

(T) A is true (resp. false) if and only if there is *something* (a truth-maker) that makes A true (resp. false).

But, again, they disagree on what this "something" is. Classically, it is reality. A judgment is made true (resp. false) by the *facts of reality*. Intuitionistically, on the other hand, this "something" is a procedure of verification, a particular *experience* of *evidence* on the part of who asserts A to be true (resp. false) in which A *reveals* itself to be true (resp. false), i.e. an experience of *recognition* of the truth condition of A. This alternative interpretation, which has important logical, mathematical and philosophical consequences, came to light, at least in mathematics, roughly a century ago in the Nederland.

In the beginning of last century, a Dutch topologist called Luitzen Egbert Jan Brouwer (known for his fixed point theorem) decided that mathematics had for its entire history been on the wrong track (more so apparently since the century before) and that it befell upon him the noble task of putting it on the right one. No ordinary mathematician would risk his/her career in making such a claim and still less in embracing such a task. But Brouwer was not an ordinary man. He bravely chose his Ph.D. thesis (*Over de grondlagen der wiskunde*, University of Amsterdam, 1907) to launch his crusade and only got the degree because his thesis advisor asked him to remove the more audacious (and outrageous) chapters from it (he didn't have any qualms in mentioning both Kant's *Critique of Pure Reason* and the *Bhagavad-Gita*).

No traditional mathematician and few philosophers would deny that logic is a fundamental, a priori science and that mathematics, or for that matter any science or rational activity, a priori or a posteriori, has to comply with its principles and laws – no one but Brouwer. For him, mathematics was primary, logic secondary. Logic, Brouwer claimed, is an a posteriori and unnecessary, but harmless, collection of principles and laws of reasoning the mathematician saw fit to apply. Truth, for Brouwer, is a matter of direct experience and principles and laws of logic are valid only to the extent they explicate the conception of truth as the *living* experience of the *adaequatio intellectus et rei* (even though neither Brouwer nor his followers have, until today, succeeded in explaining satisfactorily what should count as the *living experience of truth*, despite many attempts). Brouwer was an idealist and a mystic for whom truth was a subjective experience, not an objective fact.

One classical logical principle in particular, according to Brouwer, had to go, *tertium non datur*, which states that for *any* well-formed assertion A, either A or not-A, the negation of A, is true. For the classical logician, if A is not true, then it is false, and so the negation of A is true; for the *classical* conception of negation, not-A is true if and only if A is not true; this is what not-A *means*. For Brouwer, since one can only declare A to be true if one has *effectively* experienced the adequation of A with the facts, it may happen that A is not true, if one hasn't experienced the truth of A, but not false either, if one hasn't experienced that A *cannot* be true (i.e. that the *hypothesis* of A being true is *manifestly* false – for the intuitionist mathematician a *proof* of not-A consists in a method for effectively transforming any hypothetical proof of A into the proof of something absurd. Since a proof of an absurdity cannot exist, the proof of not-A, then, amounts to a proof that A *cannot* be proved).

For Brouwer and his followers, the intuitionists, to negate A is only justified if one has, given the established facts, *experienced* the impossibility of A; indirect evidences based on the presupposition that either A or not-A is the case cannot count. The disjunction (A or not-A), then, is true only if there are means for effectively deciding which, A or not-A, is true, even if we haven't *yet* carried out the decision procedure. If no such means are available neither is true. So, intuitionistically, that A is *not* true (i.e. that it is not the case that A is true) does *not* mean that *not*-A is true. Are they, then, both *false*? There are passages in some of Brouwer's writings ("Historical Background, Principles and Methods of Intuitionism", *South African Journal of Science* 49: 139-46, 1952, for instance) in which he explicitly says that they indeed are so, but this would have consequences for the intuitionist meaning of negation. If A and not-A could both be false, intuitionist negation would be a *contrary* rather than *contradictory* formation operator. In order to preserve negation as a contradictory formation operator both A and not-A must be divested of truth-values in case neither can be effectively verified (and for as long as they cannot be verified – as one can see, in intuitionism assertions *become* meaningful, or true, or false). Hence, for meaningful assertions to always have definite truth-values the intuitionist must deprive of meaning assertions that cannot be effectively verified. For the intuitionist, an assertion is *meaningful* if and only if it can be effectively verified, even if it hasn't yet.

If *tertium non datur* must go so must *reductio ad absurdum*, a method of proof in use since the dawn of mathematics (although criticized by Aristotle, who believed it didn't provide *explicative* proofs – even though, to this day, we still don't know what an explicative proof is), utilized, circa 300 BC, by Euclid in his *Elements* (to prove, for instance, that the diagonal of a square is incommensurable with the side, in book X), and an integral part of Archimedes' method of exhaustion. *Reductio* goes like this: to prove A, suppose not-A (in symbols $\neg$A), if we can conclude a falsity from this supposition, than we have proved $\neg\neg$A (that is, $\neg$A is *not* the case). But, since either A or $\neg$A (*tertium non datur*), and since $\neg$A is not the case, then A *must* be the case, i.e. A is proved. The truth of A follows by indirect reasoning, not direct experience (we haven't experienced A, only shown indirectly that it cannot not be true). In fact, the *law of double negation* $\neg\neg A \rightarrow A$ is equivalent, in classical or intuitionist logic, to *tertium non datur*. Now, if *tertium non datur* must go, it is adieu to the law of double negation and proofs by *reductio* too. Mathematics would then bear the consequences, since it would be deprived not only of a powerful method of proof, but of many theorems that are considered already established. This however didn't bother Brouwer in the least, for he thought we can't lose what we don't really have.

Could the unrestricted use of *tertium non datur*, besides being unjustified, lead to absurd conclusions? Kolmogoroff, Gödel and Gentzen have shown that, fortunately, this is not the case. In order to show this, let's consider Heyting's axiomatization of intuitionistically valid principles of reasoning. Although Brouwer didn't see any relevance in such an endeavor, for he didn't believe that mathematics should follow the paths prescribed a priori by logic but count only with the evidences available to the mathematician, Heyting, a disciple of Brouwer's, devised a calculus that is generally accepted as codifying intuitionistically valid rules of reasoning (Brouwer agreed, at least at the moment the system was devised, but, he thought, mathematics could very well develop others in the future. There are, however, serious doubts as to the adequacy of axiom 4 below, the so-called "principle of explosion" or *ex falso quod libet*.

Kolmogoroff thought it was not intuitionistically justifiable and *his* axiomatization of intuitionist logic does not include it. Kolmogoroff's system is equivalent to Johansson's minimal logic, which is weaker than Heyting's).

Heyting's system of intuitionist propositional calculus (IPC) has the following axioms:

(1) $A\rightarrow(B\rightarrow A)$; (2) $(A\rightarrow(B\rightarrow C))\rightarrow((A\rightarrow B)\rightarrow(A\rightarrow C))$; (3) $(A\rightarrow B)\rightarrow((A\rightarrow\neg B)\rightarrow\neg A)$
(4) $\neg A\rightarrow(A\rightarrow B)$; (5) $(A\wedge B)\rightarrow A$; (6) $(A\wedge B)\rightarrow B$; (7) $A\rightarrow(B\rightarrow(A\wedge B))$; (8) $A\rightarrow(A\vee B)$
(9) $B\rightarrow(A\vee B)$; (10) $(A\rightarrow C)\rightarrow((B\rightarrow C)\rightarrow((A\vee B)\rightarrow C))$.

The only rule of inference is *modus ponendo ponens*: from $A\rightarrow B$ and A, infer B.

In IPC we can prove the following formulas (i.e. they are theorems of the system): (11) $A\rightarrow\neg\neg A$ and (12) $\neg\neg\neg A\leftrightarrow\neg A$. However the following, which are nonetheless equivalent in IPC, cannot: (13) $\neg\neg A\rightarrow A$ (double negation) and (14) $A\vee\neg A$ (*tertium non datur*). The following however are theorems of IPC: (15) $\neg\neg(\neg\neg A\rightarrow A)$ and (16) $\neg\neg(A\rightarrow B)\rightarrow(\neg\neg A\rightarrow\neg\neg B)$.

The classical propositional calculus (CPC) is simply Heyting's calculus with either (13) or (14) added as an extra axiom. So, classical and intuitionist logics differ only with respect to the unrestricted validity of *tertium non datur* or, equivalently, the principle of double negation; the former accepts them, the latter doesn't. However, from (11) we have that the double negation of (1) – (10) is a theorem of IPC and, from (15), the same is true of (13). Moreover, from (16), if $\neg\neg(A\rightarrow B)$ and $\neg\neg A$ are theorems of IPC, then so is $\neg\neg B$ (by *modus ponens*). Therefore, the double negation of *any* theorem of CPC is a theorem of IPC. So, from an intuitionist perspective, when a classist says that A is *true* (A is provable) he's maybe only saying that *it is not the case* that A is *false* (the negation of A is refutable). Moreover, by (12), every classical theorem beginning with $\neg$ is an intuitionist theorem too, as well as any classical theorem where only $\neg$ and $\wedge$ (the connective *and*) occur.

Now, given any formula, if we substitute the symbols for $\vee$ (*or*) and $\rightarrow$ (*if...then*) by their *classical* equivalents (which, however, *nota bene*, are not their *intuitionist* equivalents): $(A\vee B)\leftrightarrow\neg(\neg A\wedge\neg B)$ and $(A\rightarrow B)\leftrightarrow\neg(A\wedge\neg B)$, we get a classically equivalent formula. Now, this tells us that we can transform systematically (Kolmogoroff-Gödel-Gentzen translation of CPC into IPC) any classical theorem in an intuitionist theorem, which proves that if CPC derived a contradiction, i.e. if it were *inconsistent*, IPC would be *too* (i.e. if CPC derived $A\wedge\neg A$, IPC would derive it too). In short, *the unrestricted use of tertium non datur or the principle of double negation cannot by themselves lead to contradictions*. IPC and CPC are equi-consistent.

It is time now we move to philosophical considerations of a different sort. Intuitionists claim that the classical conception of truth (truth as the *objective* concordance between what is said and what is the case, as opposed to the intuitionist conception of truth as the *subjective* evidence of such a concordance) in terms of which the meaning of classical logical connectives is determined and which ultimately justifies the unrestricted validity of *tertium non datur* rests on presuppositions of ontological and epistemological nature that cannot be taken for granted. On the ontological front, that the domains about which one judges exist independently (ontological realism) and are already completely determined in themselves, so that every conceivable situation in the domain is determinately either a fact or not a fact, no possible situation having an undefined ontological status (ontological determinism); on the epistemological, that judgments are decided *in themselves* (epistemological realism) and that they will be eventually decided *for us* too (epistemological optimism).

The problem seems more obvious when mathematical domains are involved. To admit that they exist independently and are fully determined in themselves looks very much like a philosophical thesis and Dummett seems to be right in saying that the acceptance of the unconditional validity of *tertium non datur* is the hallmark of realism. Can classical logic, including *tertium non datur*, be accepted in mathematics independently of realist presuppositions? After all, if mathematical "reality" does not exist in itself, if nothing is a fact that is not *made* to be a fact, mathematical facts are not *objectively* standing truth-makers. The classicist, then, seems to be facing a dilemma, either give mathematical domains independent existence and intrinsic determinacy, as intuitionists claim they do, or give up classical logic, *tertium non datur* particularly, as intuitionists say they should.

Can the classicist refuse realist allegiances and still hold on to classical logic, *y compris tertium non datur*? Let's consider the matter. The heart of the question, of course, is the theory of meaning. If Dummett's presuppositions are to be accepted, namely, that: 1) the meaning is the use, 2) meaning must be publically displayable, and 3) each individual judgment must have its own individual meaning, then his conclusions seem to impose themselves: if meaning is given by truth conditions, then we must be able to tell when truth conditions obtain when they obtain. Hence, in order to hold that meaningful judgments have truth values that are intrinsic to them independently of effective procedures of verification (but not verifications in principle, taken as ideals, as we'll see) the classicist must, or so it seems, reconsider his theory of meaning.

I want to propose one here, a theory in which the grasp of meaning does not consist in the grasp of either truth or assertability conditions, or anything for that matter, but in the correct use of largely implicit syntactic and semantic linguistic rules that constitutes linguistic competence[4].

Let's consider, to begin, the following material equivalence, with which intuitionists and classicists agree:

(V) Judgments are meaningful if and only if they are verifiable.

But whereas the intuitionist understands verifiability in an effective sense, requiring the actual existence of verification procedures, for the classicist as we'll see verifiability is only a matter of principle.

V can be used to define one leg of the equivalence in terms of the other. Intuitionists prefer to define meaningfulness by (effective) verifiability; classicists can take the alternative way, and define verifiability *in principle* by meaningfulness, giving this last notion an independent, purely linguistic characterization, one in which the grasp of the meaning of a judgment can be exhibited simply by exhibiting competence in using the rules of judgment formation. Two of the three prerequisites Dummett deemed essential for establishing intuitionist logic as the underlying logic of judgments are satisfied: meaning is giving by use and meaning is publically displayable. But it is not the case that each judgment has its own individual meaning. According to this conception, meaning is not an entity, a "something" that is attached to any individual judgment; one can be said to have grasped the meaning of a judgment only if one can be said to have mastered the language or semiotic system in which this judgment is expressed.[5]

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[4] Believing that meaning is a *thing* that meaningful judgments *posses* – isn't this an instance of what Wittgenstein detected as the origin of all philosophical pseudo-problems, language? From the fact that, we say, judgments *have* a meaning, aren't we naturally led (by language) to the conclusion that meaning is *something* judgments have?

[5] So, a parrot can utter sounds that sound like judgments, but he is not in fact expressing any for he lacks the required competence in handling the system of language.

Let me explain what this means for the atomic "S is p". Firstly, the domains of the variables "S" and "p" must be *syntactically* conform to each other, that is, "S" can only be replaced by an object-name (the name of an object) and "p" by a property-name. "The number 2 is happy" is a *syntactically* correct judgment, and hence formally consistent. Secondly, these domains must "have to do" materially with each other, that is, in the case at hand, the object must be of the type to which the property can as a matter of principle, i.e. due solely to the *meaning* attached to it, apply, even if *as a matter of fact* it doesn't. "The number 2 is happy" is not a *semantically* correct judgment, for happiness does not, as a matter of principle, befit numbers. This judgment is not *materially* consistent, but "The number 2 is odd" is both syntactically and semantically meaningful and so, I claim, it is *in principle* possible for the judging subject to experience the oddness of 2, although, as a matter of fact, he will never (and, in this case, *out of necessity*, not simply as a *contingent* fact[6]). Possibility in principle, I claim, is purely a matter of syntactic and semantic compatibility of logical and semantic *types* respectively. The *situation* (i.e. the factual *representation*) expressed by "2 is odd" can in principle be experienced simply because the type "number" and the sub-type "odd number" are obviously materially compatible[7]. A judgment is formally meaningful if their components are syntactically compatible; if they are also semantically compatible the judgment is also semantically meaningful. A meaningful judgment is one that is both syntactically and semantically meaningful. A judgment is verifiable in principle if and only if it is meaningful.

Effective verifiability is a much more stringent condition than verifiability in principle. An evidential experience is *effectively* possible only if the judging subject is in the position to bring it about if he so cared. He must know how to put himself in the position of *actually* experiencing what is effectively experienceable. This is not in general the case for experiences possible only in principle. For intuitionists, a judgment is meaningful only if it is effectively decidable, that is, if what it expresses can be effectively experienced (i.e. the content of the judgment is also the content of an effectively possible experience). Obviously, the class of intuitionistically meaningful judgments is a proper subclass of the class of classically meaningful judgments.

If S is a finite object there always is a decision procedure for no matter which "S is p", for any p; it is enough to carry out an exhaustive investigation of S to check whether it has p. Consider the judgment "there is a sequence of seven 7's in the decimal expansion of π"[8]. This judgment would be intuitionistically meaningful only if we could actually prove or disprove it, which we cannot *at the moment* (which shows that intuitionist meaning depends on time; meaningless judgments at instant t_0 can become meaningful at instant t_1 greater than t_0). Classically, however, the judgment is

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[6] There apparently is a problem here since the evenness of 2 belongs to it *by necessity*. How can a *necessarily false* judgment (2 is odd) be in principle experienced as *true*? The problem can be solved thus: the possibility in principle of experiencing as a fact that 2 is odd depends only on the fact that 2 is a number and numbers can be odd, 2 is here considered only in its most general feature, being a number; the necessary falsity of the judgment, on the other hand, involves *specific* properties of 2. So, even necessarily false judgments can in principle be true, which shows that this notion of possibility is weaker than logical possibility.

[7] I say two types are *materially compatible* if they can *in principle* (as far as their *meaning* is concerned) have at least one common instance (they may not *in fact* have one). They are *syntactically compatible* (in a judgment) if they are joined in this judgment in conformity with a priori rules of logical syntax.

[8] The finiteness or infiniteness of objects is relative to a system. All irrational numbers are infinite with respect to arbitrary *n*-ary systems of expansion (for instance, the decimal). The number π can, of course, be finitely characterized algebraically, by a rule, which despite being able to progressively generate the decimal expansion of π cannot do so in a finite amount of time.

meaningful, even though we may, as a matter of fact, be unable to verify it and so decide whether it is true or false. In finite domains, then, classically and intuitionistically meaningful judgments coincide extensionally.

Although both intuitionists and classicists agree that *tertium non datur* is valid only for meaningful judgments, they disagree, as we've seen, as to which judgments are meaningful. The consequences are obvious for the interpretation of negation. For the intuitionist, the verifiability of not-A does not follow simply from the failure to verify A. The inability of effectively experiencing A is not the same as the ability of effectively experiencing not-A. For him, to experience not-A means to experience the hypothetical experience of A as an experience of conflict or absurdity, and not being able to experience A is not the same as experiencing the impossibility of A. For the classicist, *tertium non datur* is part of the *meaning* of negation; for the intuitionist, on the other hand, it is only the expression of a property of evidencing experiences, which are *either* experiences of harmony *or* experiences of conflict, *tertium non datur*. No evidencing experience can, by *definition*, fail to be one of harmony or one of conflict.

We still have a strand to pull, the relation between meaningfulness and truth-values. Again, intuitionists and classicists alike agree that:

(P) Judgments are meaningful if and only if they possess truth-false polarity (i.e. if they have a truth value attached to them, either the true or the false).

For intuitionism and classicism alike, P follows from the fact that meaningful judgments are verifiable. But since a judgment can be classically meaningful without being intuitionistically meaningful (i.e. effectively decidable), truth values of classically but not intuitionistically meaningful judgments may be beyond effective grasp. That is, for classicists, truth is transcendent, meaningful judgments are either true or false, but not necessarily either verifiably truth or verifiably false. For the intuitionist, on the other hand, meaningful statements are *always* effectively verifiable; their truth-value is within our grasp. For him, truth is immanent.

Since, as I claim, meaningful judgments are in principle verifiability and possess truth-false polarity, but meaningfulness is not but syntactic and semantic correction, aren't I making too much follow from too little? There seems to be something missing that connects grammatical meaningfulness to intrinsic determinacy and decidability. I'll turn to this now.

Maybe the most basic difference between the classical and the intuitionist conceptions of logic is that for the former logic is a priori whereas for the latter it is a posteriori. For the intuitionist, logic depends on what is and what there is, whereas for the classicist, on the contrary, logic is independent of the facts, concerned as it is only with matters of principle (and this is why logical principles seem so "self-evident"). But consider this, even if logic does not depend on how reality is or what our powers to inspect reality are, it can very well depend on how reality is *conceived* to be, which introduces a transcendental element in our considerations. So, the classicist may claim, the validity of logical principles depends *exclusively* on how the domains about which one reasons are *meant*, their *sense of being* deriving from their intentional positing as focuses of one's cognitive interest, presupposing that sense bestowal is a prerogative of the subject.

The classicist can argue thus: logic is at the service of science and knowledge and no domain of knowledge is posited that is not conceived as *objectively*, although maybe not *independently* existing (the object of knowledge is figuratively speaking "out there" for me and anyone else who may develop an interest on it, now and for the open infinite future), completely determined *in itself* (the object of knowledge contains in itself the answer of any relevant, i.e. *meaningful* question one may raise about it) and *in* *principle* accessible to adequate experiencing (the object of knowledge offers itself unreservedly to experiencing, we are not *in principle* doomed to be forever in the dark as to any relevant, i.e. *meaningful* question we can raise about it). Science as we understand it requires no less, not as factual hypotheses, but *ideals*, over which facts have no force.

These presuppositions do not concern matters of fact, but principle, a huge difference; so, they have nothing to do with realist theses. The truth determinacy of classically meaningful judgments, i.e. the fact that they possess an intrinsic but possibly unverifiable truth-value, a truth-value in itself, as we say, is the logical counterpart of the objectivity and determinacy of the domains they refer to. *Ontological and epistemological ideals translate into logical principles*. The intrinsic truth determinacy of judgments follows from the sense bestowed on domains of knowledge, not as presuppositions of a hypothetical nature that may or not correspond to the facts, but *a priori* presuppositions required by the very idea of science.

Ontological determinacy and epistemological accessibility, moreover, are not analytic consequences of ontological realism. It does not go by itself that an independent domain is necessarily in itself already completely determined or that it is completely determinable by the knowing subject. Realism, then, contrary to what thinks Dummett, cannot by itself justify *tertium non datur*. Nothing in fact can, if justifying something is understood as the establishment of the truth of something. Principles cannot be proven, and I'm not sure they are, strictly speaking, true either, much less logical principles. But they can be justified, in a different sense of justification, as consequences of ideals required by the very idea of science. Domains of knowledge are posited as intrinsically determined (even though maybe not independently existing) and in principle accessible to verifying experiences for otherwise science would be an impossible project. This seems to me a perfectly legitimate justification of logical principles, *tertium non datur* in particular.

Bibliography

Brouwer, L. E. J. "Intuitionist Splitting of the Fundamental Notions of Mathematics". In P. Mancosu 1998: 286-89 (1923), 290-92 (1925), see also the addendum, 293-95 (1927).

Burgess, John P. *Philosophical Logic*. Princeton and Oxford: Princeton University Press, 2009.

Dummett, Michael. "The Philosophical Basis of Intuitionistic Logic". In his *Truth and Other Enigmas* (2nd ed.), Cambridge, Mass.: Harvard University Press, 1980 (1978).

Haack, Susan. *Philosophies of Logic*. Cambridge: Cambridge University Press, 1978.

_______. *Deviant Logic, Fuzzy Logic – Beyond the Formalism*. Chicago and London: The University of Chicago Press, 1996 (1974).

Martin-Löf, Per. "On the Meaning of the Logical Constants and the Justifications of the Logical Laws". *Nordic Journal of Philosophical Logic* 1(1): 11-60, 1996.

Mancosu, Paolo. *From Brouwer to Hilbert, the Debate on the Foundations of Mathematics in the 1920's*. New York and Oxford: Oxford University Press, 1998.